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Elastodynamic Green's functions for orthotropic plane strain continua with inclined axes of symmetry

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Abstract

The article reports a methodology to synthesize the response of plane strain orthotropic and transversely isotropic half-spaces and full-spaces with arbitrarily oriented symmetry axes and subjected to concentrated and to distributed loads. Numerical results include the response of a half-space and a full-space to a uniform strip load. The examples presented analyze the influence of coordinate axis rotation and of the continuum anisotropy ratios. Stationary dynamic behavior is assumed throughout the article.

1. Introduction

In many engineering phenomena, including the response of soils, geological materials and composites, the assumption of an isotropic behavior may not capture some significant features of the continuum response. The formulation and solution of anisotropic problems is far more difficult and cumbersome that its isotropic counterpart. In the last years the elastodynamic response of anisotropic continuum has received the attention of several researchers. In particular transversely isotropic and orthotropic materials, which may not be distinguished from each other in plane strain and plane stress cases, have been more regularly studied.

A review of the literature on wave propagation in anisotropic continua shows that Carrier (1946) analyzed waves in transversely isotropic media subjected to a restriction in the constitutive equations which would allow the classical Helmholtz decomposition of a vector field to be applied. Later Stoneley (1949), Synge (1956) and Buchwald (1961) extended the analysis to handle more general anisotropic material behavior. The synthesis of dynamic Green's functions for anisotropic media has only been accomplished more recently. In 1975 Payton presented a time domain solution for displacements and stresses in a transversely isotropic full-space, subjected to Carrier's restriction

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and loaded by an instantaneously applied point force. In his book, published in 1983, the same solution without Carrier's restriction was presented. The book also contains the solution for a point force instantaneously applied on the surface of a transversely isotropic half-space. Frequency domain Green's functions, for harmonically applied time loads, have been developed in this decade by Rajapakse and Wang (1991, 1993). In particular, the last two articles report the solutions for loads applied within a 2- or 3-D transversely isotropic half-space. Three-dimensional time harmonic Green's functions for a transversely isotropic media as a double integral representation over a finite domain was given by Zhu (1992). A procedure to synthesize 2-D Green's functions for general anisotropic media using a Fourier integral transform and a modal expansion is furnished by Liu and Lam (1996).

A common feature of the mentioned work is the supposition that the symmetry axes for the transversely isotropic continuum or the principal direction axes for the orthotropic medium would be coincidental with the chosen coordinate system axes. In solving half-space problems, possessing a free surface, the usual procedure is to choose the coordinate axes parallel and perpendicular to the free surface. But in the general case—either 2- or 3-D—the medium principal or symmetry axes are not necessarily parallel or perpendicular to the half-space surface. Solutions which enforce this hypothesis must be regarded as particular cases.

The present article reports the synthesis of Green's and influence functions, i.e., the solutions for point and distributed loads applied on the surface of a half-space and in the interior of a full-space for the case that the symmetry or principal axes of the transversely isotropic medium are arbitrarily oriented. A state of plane strain and harmonic time dependence is assumed throughout the article. Numerical results are presented for distributed loads acting on the surface of the half-space and in the interior of the full-space, for distinct anisotropy ratios and for various inclinations of the continuum principal axes. Whenever possible comparisons are made with results obtained by other authors and methodologies.

2. Problem formulation

2.1. Statement of the problem

Consider an elastic orthotropic half-space and a coordinate system xyz describing the principal directions of the continuum, as shown in Fig. 1. Consider also a second coordinate system $x'y'z'$ obtained by rotating the first system about the y-axis by an angle θ in such a way that the axis x' is parallel to the half-space surface. On the half-space surface time harmonic traction loads are applied $p(x', t) = p(x')\exp(\mathrm{i}\omega t)$ with ω being the circular frequency and $\mathrm{i} = \sqrt{-1}$.

2.2. Constitutive equations

The stress–strain relations for an orthotropic material considering plane strain rate are given by (Lekhnitskii, 1981):

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 \\ c_{13} & c_{33} & 0 \\ 0 & 0 & 2c_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \varepsilon_{xz} \end{Bmatrix} \tag{1}$$

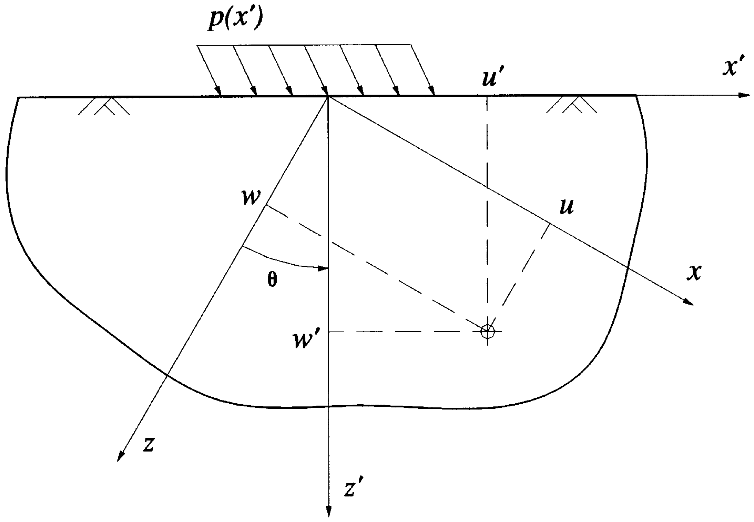

Fig. 1. Load applied on the surface of a half-space.

where c_{11}, c_{13}, c_{33} and c_{44} are the continuum elastic constants. The isotropic elastic material can be regarded as a particular case with $c_{11} = c_{33}$ and $c_{44} = (c_{11} - c_{13})/2$. These relations may be used to create two dimensionless indexes n_1, n_3 that express the degree of the material anisotropy (Anderson, 1961):

$$\begin{aligned} n_1 &= c_{33}/c_{11} \\ n_3 &= (c_{11} - 2c_{44})/c_{13} \end{aligned} \tag{2}$$

For a linear elastic anisotropic medium the fourth-order tensor containing the material constants c_{ijkl} does depend on the coordinate system in which the constitutive equations are written. Defining the direction cosines α_{ij} between the i'-th-direction in the $x'y'z'$-coordinate system and the j-th-direction in the xyz-coordinate system, the material constants in the primed (rotated) system may be obtained through the relation

$$c'_{ijkl} = \alpha_{ip}\alpha_{jq}\alpha_{kr}\alpha_{ls}c_{pqrs} \tag{3}$$

For the rotated coordinate system $x'y'z'$ the constitutive equations (1) relating stresses $\sigma_{i'j'}$ and linear strains $\varepsilon_{i'j'}$ are given by:

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{z'z'} \\ \sigma_{x'z'} \end{Bmatrix} = \begin{bmatrix} c'_{11} & c'_{13} & c'_{15} \\ c'_{13} & c'_{33} & c'_{35} \\ c'_{15} & c'_{35} & c'_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{z'z'} \\ 2\varepsilon_{x'z'} \end{Bmatrix} \tag{4}$$

An analysis of the previous equations indicate that in the rotated system there is a coupling between shear stresses $\sigma_{x'z'}$ and normal strains $\varepsilon_{x'x'}$, $\varepsilon_{z'z'}$.

2.3. Governing equations

In the absence of body forces, the equations governing the displacements $u(x,z)$, $w(x,z)$ of the orthotropic plane strain continuum in the time domain are (Payton, 1983):

$$c_{11}\frac{\partial^2 u}{\partial x^2}+c_{44}\frac{\partial^2 u}{\partial z^2}+(c_{13}+c_{44})\frac{\partial^2 w}{\partial x\,\partial z}=\rho\frac{\partial^2 u}{\partial t^2}$$

$$c_{44}\frac{\partial^2 w}{\partial x^2}+c_{33}\frac{\partial^2 w}{\partial z^2}+(c_{13}+c_{44})\frac{\partial^2 u}{\partial x\,\partial z}=\rho\frac{\partial^2 w}{\partial t^2} \tag{5}$$

Assuming time harmonic behavior $u(x,z,t)=u(x,z)\,\mathrm{e}^{\mathrm{i}\omega t}$, $w(x,z,t)=w(x,z)\,\mathrm{e}^{\mathrm{i}\omega t}$ eqn (5) may be recast into:

$$\beta\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}+\kappa\frac{\partial^2 w}{\partial x\,\partial z}=-\delta^2 u$$

$$\frac{\partial^2 w}{\partial x^2}+\alpha\frac{\partial^2 w}{\partial z^2}+\kappa\frac{\partial^2 u}{\partial x\,\partial z}=-\delta^2 w \tag{6}$$

In eqn (6) the following dimensionless elastic constants were introduced:

$$\alpha=c_{33}/c_{44}$$

$$\beta=c_{11}/c_{44}$$

$$\kappa=(c_{13}+c_{44})/c_{44} \tag{7}$$

A normalized frequency parameter δ has also been introduced in (6) and is defined by:

$$\delta^2=\rho\omega^2/c_{44} \tag{8}$$

With the aid of expressions relating to the displacements in the rotated system $u'(x',z')$, $w'(x',z')$ to the displacements in the original system $u(x,z)$, $w(x,z)$

$$u'=u\cos\theta-w\sin\theta$$

$$w'=u\sin\theta+w\cos\theta \tag{9}$$

the equations of motion (6) may be written in the primed coordinate system:

$$\frac{\partial^2 u'}{\partial x'^2}K_{uxx}+\frac{\partial^2 u'}{\partial x'\,\partial z'}K_{uxz}+\frac{\partial^2 u'}{\partial z'^2}K_{uzz}+\frac{\partial^2 w'}{\partial x'^2}K_{wxx}+\frac{\partial^2 w'}{\partial x'\,\partial z'}K_{wxz}$$

$$+\frac{\partial^2 w'}{\partial z'^2}K_{wzz}+\delta^2(u'\cos\theta+w'\sin\theta)=0$$

$$-\frac{\partial^2 u'}{\partial x'^2}L_{uxx}+\frac{\partial^2 u'}{\partial x'\,\partial z'}L_{uxz}-\frac{\partial^2 u'}{\partial z'^2}L_{uzz}+\frac{\partial^2 w'}{\partial x'^2}L_{wxx}-\frac{\partial^2 w'}{\partial x'\,\partial z'}L_{wxz}$$

$$+\frac{\partial^2 w'}{\partial z'^2} L_{wzz} + \delta^2(-u' \sin\theta + w' \cos\theta) = 0 \tag{10}$$

The constants K_{ijk} and L_{ijk} depend on the normalized material constants α, β, κ, on the rotation angle θ and are given by:

$$\begin{aligned}
K_{uxx} &= \beta \cos^3\theta + (\kappa+1)\sin^2\theta\cos\theta \\
K_{uxz} &= 2(\beta-1)\sin\theta\cos^2\theta - \kappa(\cos^2\theta - \sin^2\theta)\sin\theta \\
K_{uzz} &= (\beta-\kappa)\sin^2\theta\cos\theta + \cos^3\theta \\
K_{wxx} &= (\beta-\kappa)\sin\theta\cos^2\theta + \sin^3\theta \\
K_{wxz} &= 2(\beta-1)\sin^2\theta\cos\theta + \kappa(\cos^2\theta - \sin^2\theta)\cos\theta \\
K_{wzz} &= \beta\sin^3\theta + (\kappa+1)\sin\theta\cos^2\theta
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
L_{uxx} &= \alpha\sin^3\theta + (\kappa+1)\sin\theta\cos^2\theta \\
L_{uxz} &= 2(\alpha-1)\sin^2\theta\cos\theta + \kappa(\cos^2\theta - \sin^2\theta)\cos\theta \\
L_{uzz} &= (\alpha-\kappa)\sin\theta\cos^2\theta + \sin^3\theta \\
L_{wxx} &= (\alpha-\kappa)\sin^2\theta\cos\theta + \cos^3\theta \\
L_{wxz} &= 2(\alpha-1)\sin^2\theta\cos^2\theta - \kappa(\cos^2\theta - \sin^2\theta)\sin\theta \\
L_{wzz} &= \alpha\cos^3\theta + (\kappa+1)\sin^2\theta\cos^2\theta
\end{aligned} \tag{12}$$

The Fourier integral transform relating the coordinate x' to the wave number variable λ and its inverse are defined, respectively, by (Sneddon, 1951):

$$\bar{f}(\lambda, z') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x', z')\, e^{-i\lambda x'}\, dx', \quad f(x', z') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\lambda, z')\, e^{i\lambda x'}\, d\lambda \tag{13}$$

Introducing the dimensionless parameter $\zeta = \lambda/\delta$, the fourth-order uncoupled Fourier transformed version of eqns (10) in the dimensionless wave number domain are (Wylie and Barrett, 1985):

$$\begin{aligned}
&\delta^4(T_1\zeta^4 - T_2\zeta^2 + 1)\bar{u}' + i\delta^3(T_3\zeta^3 - T_4\zeta)\frac{\partial \bar{u}'}{\partial z'} - \delta^2(T_5\zeta^2 - T_6)\frac{\partial^2 \bar{u}'}{\partial z'^2} + iT_7\delta\zeta\frac{\partial^3 \bar{u}'}{\partial z'^3} + T_8\frac{\partial^4 \bar{u}'}{\partial z'^4} = 0 \\
&\delta^4(T_1\zeta^4 - T_2\zeta^2 + 1)\bar{w}' + i\delta^3(T_3\zeta^3 - T_4\zeta)\frac{\partial \bar{w}'}{\partial z'} - \delta^2(T_5\zeta^2 - T_6)\frac{\partial^2 \bar{w}'}{\partial z'^2} + iT_7\delta\zeta\frac{\partial^3 \bar{w}'}{\partial z'^3} + T_8\frac{\partial^4 \bar{w}'}{\partial z'^4} = 0
\end{aligned} \tag{14}$$

The constants T_i $(i = 1, \ldots, 8)$ shown in eqns (14) are given by:

$$
\begin{aligned}
T_1 &= \tfrac{1}{8}[3(\alpha+\beta)+\gamma-4(\alpha-\beta)\cos 2\theta+(\alpha+\beta-\gamma)\cos 4\theta]\\
T_2 &= 1+\tfrac{1}{2}[\alpha+\beta-(\alpha-\beta)\cos 2\theta]\\
T_3 &= [\alpha-\beta-(\alpha+\beta-\gamma)\cos 2\theta]\sin 2\theta\\
T_4 &= (\alpha-\beta)\sin 2\theta\\
T_5 &= \tfrac{1}{4}[3(\alpha+\beta)+\gamma-3(\alpha+\beta-\gamma)\cos 4\theta]\\
T_6 &= 1+\tfrac{1}{2}[\alpha+\beta+(\alpha-\beta)\cos 2\theta]\\
T_7 &= [-\alpha+\beta-(\alpha+\beta-\gamma)\cos 2\theta]\sin 2\theta\\
T_8 &= \tfrac{1}{8}[3(\alpha+\beta)+\gamma+4(\alpha-\beta)\cos 2\theta+(\alpha+\beta-\gamma)\cos 4\theta]
\end{aligned}
\tag{15}
$$

These terms depend on the normalized elastic constants α, β, the rotation angle θ and on the parameter γ defines as:

$$\gamma = 1+\alpha\beta-\kappa^2 \tag{16}$$

2.4. Solution for displacements and stresses

The general solutions for the ordinary differential equations (14) in the transformed domain (λ, z') are (Wylie and Barrett, 1985):

$$
\begin{aligned}
\bar{w}(\lambda, z') &= A\,e^{\delta\xi_1 z'}+B\,e^{\delta\xi_2 z'}+C\,e^{\delta\xi_3 z'}+D\,e^{\delta\xi_4 z'}\\
\bar{u}(\lambda, z') &= A\omega_1'\,e^{\delta\xi_1 z'}+B\omega_2'\,e^{\delta\xi_2 z'}+C\omega_3'\,e^{\delta\xi_3 z'}+D\omega_4'\,e^{\delta\xi_4 z'}
\end{aligned}
\tag{17}
$$

where

$$\omega_i' = \frac{\zeta^2 L_{wxx}+i\zeta L_{wxz}\xi_i-L_{wzz}\xi_i^2-\cos\theta}{\zeta^2 L_{uxx}+i\zeta L_{uxz}\xi_i-L_{uzz}\xi_i^2-\sin\theta}, \quad i = 1,\ldots,4 \tag{18}$$

and ξ_i, $i = 1, \ldots, 4$ are the roots of the polynomial equation:

$$T_1\zeta^4-T_2\zeta^2+1+i\xi(T_3\zeta^3-T_4\zeta)-\xi^2(T_5\zeta^2-T_6)+iT_7\zeta\xi^3+T_8\xi^4 = 0 \tag{19}$$

Once the displacement solutions are known the stress components may be determined with the help of eqn (4) yielding:

$$
\begin{aligned}
\bar{\sigma}_{x'x'} &= \delta c_{44}(f_1 A\,e^{\delta\xi_1 z'}+f_2 B\,e^{\delta\xi_2 z'}+f_3 C\,e^{\delta\xi_3 z'}+f_4 D\,e^{\delta\xi_4 z'})\\
\bar{\sigma}_{z'z'} &= \delta c_{44}(g_1 A\,e^{\delta\xi_1 z'}+g_2 B\,e^{\delta\xi_2 z'}+g_3 C\,e^{\delta\xi_3 z'}+g_4 D\,e^{\delta\xi_4 z'})\\
\bar{\sigma}_{x'z'} &= \delta c_{44}(h_1 A\,e^{\delta\xi_1 z'}+h_2 B\,e^{\delta\xi_2 z'}+h_3 C\,e^{\delta\xi_3 z'}+h_4 D\,e^{\delta\xi_4 z'})
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
f_i &= i\zeta\bar{c}_{11}'\omega_i'+\bar{c}_{13}'\xi_i+\bar{c}_{15}'(i\zeta+\xi_i\omega_i)\\
g_i &= i\zeta\bar{c}_{13}'\omega_i'+\bar{c}_{33}'\xi_i+\bar{c}_{35}'(i\zeta+\xi_i\omega_i)
\end{aligned}
$$

$$h_i = \mathrm{i}\zeta \bar{c}'_{15}\omega'_i + \bar{c}'_{35}\xi_i + \bar{c}'_{55}(\mathrm{i}\zeta + \xi_i\omega_i), \quad i = 1, \ldots, 4 \tag{21}$$

and the normalized dimensionless material constants $\bar{c}'_{ij}$ are given by:

$$\bar{c}'_{11} = \frac{c'_{11}}{c_{44}} = \beta\cos^4\theta + \alpha\sin^4\theta + 2(\kappa+1)\sin^2\theta\cos^2\theta$$

$$\bar{c}'_{13} = \frac{c'_{13}}{c_{44}} = (\beta+\alpha-4)\sin^2\theta\cos^2\theta + (\kappa-1)(\cos^4\theta + \sin^4\theta)$$

$$\bar{c}'_{15} = \frac{c'_{15}}{c_{44}} = (\kappa+1-\alpha)\sin^3\theta\cos\theta - (\kappa+1-\beta)\sin\theta\cos^3\theta$$

$$\bar{c}'_{33} = \frac{c'_{33}}{c_{44}} = \alpha\cos^4\theta + \beta\sin^4\theta + 2(\kappa+1)\sin^2\theta\cos^2\theta$$

$$\bar{c}'_{35} = \frac{c'_{35}}{c_{44}} = (\kappa+1-\alpha)\sin\theta\cos^3\theta - (\kappa+1-\beta)\sin^3\theta\cos\theta$$

$$\bar{c}'_{55} = \frac{c'_{55}}{c_{44}} = [\beta+\alpha-2(\kappa-1)]\sin^2\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)^2 \tag{22}$$

3. Synthesis of Green's functions and influence functions

In this section the general solutions presented for displacements (17) and stresses (20) will be particularized to yield the response of a half-space and a full-space subjected to a point load and to a uniform traction distribution. The loads applied at an arbitrary direction will be decomposed according to the axes $x'z'$ and the corresponding solutions will be superposed. The constants A, B, C, D will be determined by imposing the proper boundary conditions for each stress boundary value problem. These constants when substituted in (17) and (20) deliver the displacement and stress solutions in the Fourier transformed (ζ, z') domain. The final solution in the original domain (x', z') is obtained by a numerical inversion of the Fourier integral transform.

3.1. Loads on the surface of a half-space

Consider an elastic orthotropic half-space domain defined by:

$$|x'| < \infty, \quad 0 \leqslant z' < \infty \tag{23}$$

The principal (symmetry) directions of the medium are rotated with respect to the $x'z'$-coordinate system by an angle θ (Fig. 1). On the surface of the half-space ($z' = 0$) time harmonic loads are applied in the horizontal (x') and in the vertical direction (z').

(1) First stress boundary value problem. Loads in the direction x':
For this loading case the (stress) boundary conditions are:

$$\begin{aligned} &\sigma_{x'z'}(x', z' = 0) = -p_{x'}(x') \\ &\sigma_{z'z'}(x', z' = 0) = 0 \end{aligned} \tag{24}$$

(2) Second stress boundary value problem. Loads in the direction z':

For this loading case the (stress) boundary conditions are:

$$\begin{aligned} &\sigma_{x'z'}(x', z' = 0) = 0 \\ &\sigma_{z'z'}(x', z' = 0) = -p_{z'}(x') \end{aligned} \tag{25}$$

It can be seen that there are four unknown constants (A, B, C, D) to be determined in eqns (17) and (20) and only two boundary conditions in each case. But it should be noticed that besides satisfying the boundary conditions (24) or (25) the solutions must also respect the radiation condition. For the half-space the radiations condition implies outgoing waves with decreasing amplitudes in the positive z'-direction. An analysis of the structure of the eqn (19) shows that if all T_i are real and ξ_k is one of the roots then the negative of its conjugate $-\bar{\xi}_k$ is another root. This implies that two of the four roots of eqn (19) do not fulfil the radiation condition and must, therefore, vanish (Barros, 1997). These roots will be associated with the constants C and D, consequently the radiation condition imposes that $C(\zeta) = D(\zeta) = 0$.

3.1.1. Green's functions

To synthesize the Green's functions, i.e., the displacement and stress solutions due to a point load described as a Dirac's Delta $p_{x'}(x') = p_{z'}(x') = \delta_D(x')$, its Fourier transform with respect to the pair (x', λ)

$$\bar{p}_{x'}(\lambda) = \bar{p}_{z'}(\lambda) = \frac{1}{\sqrt{2\pi}} \tag{26}$$

must be used. The final Green's functions in the (x', z') domain is obtained by a numerical inversion of the Fourier integral transform. The expressions for displacements and stresses are, respectively:

$$\begin{aligned} G_{x'x'} &= \frac{1}{2\pi c_{44}} \int_{-\infty}^{\infty} \frac{1}{g_1 h_2 - g_2 h_1} (g_2 \omega'_1 \, \mathrm{e}^{\delta \xi_1 z'} - g_1 \omega'_2 \, \mathrm{e}^{\delta \xi_2 z'}) \, \mathrm{e}^{\mathrm{i}\delta \zeta x'} \, \mathrm{d}\zeta \\ G_{z'x'} &= \frac{1}{2\pi c_{44}} \int_{-\infty}^{\infty} \frac{1}{g_1 h_2 - g_2 h_1} (g_2 \, \mathrm{e}^{\delta \xi_1 z'} - g_1 \, \mathrm{e}^{\delta \xi_2 z'}) \, \mathrm{e}^{\mathrm{i}\delta \zeta x'} \, \mathrm{d}\zeta \\ G_{x'z'} &= -\frac{1}{2\pi c_{44}} \int_{-\infty}^{\infty} \frac{1}{g_1 h_2 - g_2 h_1} (h_2 \omega'_1 \, \mathrm{e}^{\delta \xi_1 z'} - h_1 \omega'_2 \, \mathrm{e}^{\delta \xi_2 z'}) \, \mathrm{e}^{\mathrm{i}\delta \zeta x'} \, \mathrm{d}\zeta \\ G_{z'z'} &= -\frac{1}{2\pi c_{44}} \int_{-\infty}^{\infty} \frac{1}{g_1 h_2 - g_2 h_1} (h_2 \, \mathrm{e}^{\delta \xi_1 z'} - h_1 \, \mathrm{e}^{\delta \xi_2 z'}) \, \mathrm{e}^{\mathrm{i}\delta \zeta x'} \, \mathrm{d}\zeta \end{aligned} \tag{27}$$

and

$$\sigma_{x'x'x'} = \frac{\delta}{2\pi}\int_{-\infty}^{\infty} \frac{1}{g_1h_2-g_2h_1}(f_1g_2\,\mathrm{e}^{\delta\xi_1z'}-f_2g_1\,\mathrm{e}^{\delta\xi_2z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$\sigma_{x'z'x'} = \frac{\delta}{2\pi}\int_{-\infty}^{\infty} \frac{1}{g_1h_2-g_2h_1}(h_1g_2\,\mathrm{e}^{\delta\xi_1z'}-h_2g_1\,\mathrm{e}^{\delta\xi_2z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$\sigma_{z'z'x'} = \frac{\delta}{2\pi}\int_{-\infty}^{\infty} \frac{g_1g_2}{g_1h_2-g_2h_1}(\mathrm{e}^{\delta\xi_1z'}-\mathrm{e}^{\delta\xi_2z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$\sigma_{x'x'z'} = -\frac{\delta}{2\pi}\int_{-\infty}^{\infty} \frac{1}{g_1h_2-g_2h_1}(f_1h_2\,\mathrm{e}^{\delta\xi_1z'}-f_2h_1\,\mathrm{e}^{\delta\xi_2z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$\sigma_{x'z'z'} = -\frac{\delta}{2\pi}\int_{-\infty}^{\infty} \frac{h_1h_2}{g_1h_2-g_2h_1}(\mathrm{e}^{\delta\xi_1z'}-\mathrm{e}^{\delta\xi_2z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$\sigma_{z'z'z'} = -\frac{\delta}{2\pi}\int_{-\infty}^{\infty} \frac{1}{g_1h_2-g_2h_1}(g_1h_2\,\mathrm{e}^{\delta\xi_1z'}-g_2h_1\,\mathrm{e}^{\delta\xi_2z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta \tag{28}$$

3.1.2. Influence functions

There are two possible ways to obtain the so-called half-space influence functions, i.e., the solutions due to distributed loads applied at the half-space surface. The first one is to convolute the distributed load with the point load solution. The second approach, which is used in this article, is to set directly the distributed loads $p_{x'}(x')$ or $p_{z'}(x')$ in eqns (24) or (25). The Fourier transform with respect to the pair (x', λ) for the case of a uniform strip load of amplitude p_0 and width $2a$ applied at the origin of the coordinate system ($x' = z' = 0$) is:

$$\bar{p}_i(\lambda) = p_{0i}\frac{2\sin(\lambda a)}{\sqrt{2\pi}\lambda}, \quad i = x', z' \tag{29}$$

where $p_{0x'}$ and $p_{0z'}$ are, respectively, the components of p_0 in the horizontal and vertical directions. The solutions for displacements due to the uniformly distributed loads of unit intensity ($p_{0i} = 1$), are:

$$u'_{cx'} = \frac{1}{\pi c_{44}\delta}\int_{-\infty}^{\infty} \frac{1}{R'_c}(g_2\omega'_1\,\mathrm{e}^{\delta\xi_1z'}-g_1\omega'_2\,\mathrm{e}^{\delta\xi_2z'})\,\mathrm{e}^{-\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$w'_{cx'} = \frac{1}{\pi c_{44}\delta}\int_{-\infty}^{\infty} \frac{1}{R'_c}(g_2\,\mathrm{e}^{\delta\xi_1z'}-g_1\,\mathrm{e}^{\delta\xi_2z'})\,\mathrm{e}^{-\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$u'_{cz'} = -\frac{1}{\pi c_{44}\delta}\int_{-\infty}^{\infty} \frac{1}{R'_c}(h_2\omega'_1\,\mathrm{e}^{\delta\xi_1z'}-h_1\omega'_2\,\mathrm{e}^{\delta\xi_2z'})\,\mathrm{e}^{-\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$w'_{cz'} = -\frac{1}{\pi c_{44}\delta}\int_{-\infty}^{\infty} \frac{1}{R'_c}(h_2\,\mathrm{e}^{\delta\xi_1z'}-h_1\,\mathrm{e}^{\delta\xi_2z'})\,\mathrm{e}^{-\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta \tag{30}$$

with

$$R'_c = \frac{\zeta(g_1 h_2 - g_2 h_1)}{\sin(\delta\zeta a)} \tag{31}$$

The solutions for stresses are obtained analogously (Barros, 1997).

3.2. Loads applied in the interior of the full-space

To analyze displacements and stresses at the interior of a full-space due to an harmonic point load, the continuum is divided in two half-spaces defined by:

- Half-space 1: $|x'| < \infty,\ -\infty < z' \leqslant 0$.
- Half-space 2: $|x'| < \infty,\ 0 \leqslant z' < \infty$.

(1) Load in the x'-direction:

The boundary conditions for a load $p_{x'}$ applied at the interface of the two half-spaces ($z' = 0$) in the x'-direction are:

$$\begin{aligned}
&u'^{(1)}(x',0) - u'^{(2)}(x',0) = 0\\
&w'^{(1)}(x',0) - w'^{(2)}(x',0) = 0\\
&\sigma^{(1)}_{x'z'}(x',0) - \sigma^{(2)}_{x'z'}(x,0) = p_{x'}(x')\\
&\sigma^{(1)}_{z'z'}(x,0) - \sigma^{(2)}_{z'z'}(x',0) = 0
\end{aligned} \tag{32}$$

(2) Load in the z'-direction:

Analogously the boundary conditions for a load $p_{z'}$ applied at the interface of the two half-spaces ($z' = 0$) in the z'-direction are:

$$\begin{aligned}
&u'^{(1)}(x',0) - u'^{(2)}(x',0) = 0\\
&w'^{(1)}(x',0) - w'^{(2)}(x',0) = 0\\
&\sigma^{(1)}_{x'z'}(x',0) - \sigma^{(2)}_{x'z'}(x,0) = 0\\
&\sigma^{(1)}_{z'z'}(x,0) - \sigma^{(2)}_{z'z'}(x',0) = p_{z'}(x')
\end{aligned} \tag{33}$$

3.2.1. Green's functions

There are four unknown constants for each half-space $A^{(i)} = B^{(i)} = C^{(i)} = D^{(i)}$, $i = 1, 2$ and only four boundary conditions for each stress boundary value problem (32) and (33). Therefore, four constants must vanish due to the radiation condition. These are assumed to be $A^{(1)} = B^{(1)} = C^{(2)} = D^{(2)} = 0$. The resulting displacements and stresses are, respectively:

1. Half-space 1 ($z' \leqslant 0$):

$$G_{x'x'} = \frac{1}{\sqrt{2\pi}c_{44}} \int_{-\infty}^{\infty} \frac{\bar{p}_{x'}}{H} (C_{x'}\omega'_3\, e^{\delta\xi_3 z'} + D_{x'}\omega'_4\, e^{\delta\xi_4 z'})\, e^{i\delta\zeta x'}\, d\zeta$$

$$G_{z'x'} = \frac{1}{\sqrt{2\pi}c_{44}} \int_{-\infty}^{\infty} \frac{\bar{p}_{x'}}{H}(C_{x'}\,\mathrm{e}^{\delta\xi_3 z'} + D_{x'}\,\mathrm{e}^{\delta\xi_4 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$G_{x'z'} = \frac{1}{\sqrt{2\pi}c_{44}} \int_{-\infty}^{\infty} \frac{\bar{p}_{z'}}{H}(C_{z'}\omega'_3\,\mathrm{e}^{\delta\xi_3 z'} + D_{z'}\omega'_4\,\mathrm{e}^{\delta\xi_4 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$G_{z'z'} = \frac{1}{\sqrt{2\pi}c_{44}} \int_{-\infty}^{\infty} \frac{\bar{p}_{z'}}{H}(C_{z'}\,\mathrm{e}^{\delta\xi_3 z'} + D_{z'}\,\mathrm{e}^{\delta\xi_4 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta \tag{34}$$

and

$$\sigma_{x'x'x'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{x'}}{H}(C_{x'}f_3\,\mathrm{e}^{\delta\xi_3 z'} + D_{x'}f_4\,\mathrm{e}^{\delta\xi_4 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$\sigma_{x'z'x'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{x'}}{H}(C_{x'}h_3\,\mathrm{e}^{\delta\xi_3 z'} + D_{x'}h_4\,\mathrm{e}^{\delta\xi_4 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$\sigma_{z'z'x'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{x'}}{H}(C_{x'}g_3\,\mathrm{e}^{\delta\xi_3 z'} + D_{x'}g_4\,\mathrm{e}^{\delta\xi_4 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$\sigma_{x'x'z'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{z'}}{H}(C_{z'}f_3\,\mathrm{e}^{\delta\xi_3 z'} + D_{z'}f_4\,\mathrm{e}^{\delta\xi_4 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$\sigma_{x'z'z'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{z'}}{H}(C_{z'}h_3\,\mathrm{e}^{\delta\xi_3 z'} + D_{z'}h_4\,\mathrm{e}^{\delta\xi_4 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$\sigma_{z'z'z'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{z'}}{H}(C_{z'}g_3\,\mathrm{e}^{\delta\xi_3 z'} + D_{z'}g_4\,\mathrm{e}^{\delta\xi_4 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta \tag{35}$$

2. Half-space 2 ($z' \geqslant 0$):

$$G_{x'x'} = \frac{1}{\sqrt{2\pi}c_{44}} \int_{-\infty}^{\infty} \frac{\bar{p}_{x'}}{H}(A_{x'}\omega'_1\,\mathrm{e}^{\delta\xi_1 z'} + B_{x'}\omega'_2\,\mathrm{e}^{\delta\xi_2 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$G_{z'x'} = \frac{1}{\sqrt{2\pi}c_{44}} \int_{-\infty}^{\infty} \frac{\bar{p}_{x'}}{H}(A_{x'}\,\mathrm{e}^{\delta\xi_1 z'} + B_{x'}\,\mathrm{e}^{\delta\xi_2 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$G_{x'z'} = \frac{1}{\sqrt{2\pi}c_{44}} \int_{-\infty}^{\infty} \frac{\bar{p}_{z'}}{H}(A_{z'}\omega'_1\,\mathrm{e}^{\delta\xi_1 z'} + B_{z'}\omega'_2\,\mathrm{e}^{\delta\xi_2 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta$$

$$G_{z'z'} = \frac{1}{\sqrt{2\pi}c_{44}} \int_{-\infty}^{\infty} \frac{\bar{p}_{z'}}{H}(A_{z'}\,\mathrm{e}^{\delta\xi_1 z'} + B_{z'}\,\mathrm{e}^{\delta\xi_2 z'})\,\mathrm{e}^{\mathrm{i}\delta\zeta x'}\,\mathrm{d}\zeta \tag{36}$$

and

$$\sigma_{x'x'x'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{x'}}{H} (A_{x'} f_1 \, \mathrm{e}^{\delta \xi_1 z'} + B_{x'} f_2 \, \mathrm{e}^{\delta \xi_2 z'}) \, \mathrm{e}^{\mathrm{i}\delta\zeta x'} \, \mathrm{d}\zeta$$

$$\sigma_{x'z'x'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{x'}}{H} (A_{x'} h_1 \, \mathrm{e}^{\delta \xi_1 z'} + B_{x'} h_2 \, \mathrm{e}^{\delta \xi_2 z'}) \, \mathrm{e}^{\mathrm{i}\delta\zeta x'} \, \mathrm{d}\zeta$$

$$\sigma_{z'z'x'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{x'}}{H} (A_{x'} g_1 \, \mathrm{e}^{\delta \xi_1 z'} + B_{x'} g_2 \, \mathrm{e}^{\delta \xi_2 z'}) \, \mathrm{e}^{\mathrm{i}\delta\zeta x'} \, \mathrm{d}\zeta$$

$$\sigma_{x'x'z'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{z'}}{H} (A_{z'} f_1 \, \mathrm{e}^{\delta \xi_1 z'} + B_{z'} f_2 \, \mathrm{e}^{\delta \xi_2 z'}) \, \mathrm{e}^{\mathrm{i}\delta\zeta x'} \, \mathrm{d}\zeta$$

$$\sigma_{x'z'z'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{z'}}{H} (A_{z'} h_1 \, \mathrm{e}^{\delta \xi_1 z'} + B_{z'} h_2 \, \mathrm{e}^{\delta \xi_2 z'}) \, \mathrm{e}^{\mathrm{i}\delta\zeta x'} \, \mathrm{d}\zeta$$

$$\sigma_{z'z'z'} = \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{p}_{z'}}{H} (A_{z'} g_1 \, \mathrm{e}^{\delta \xi_1 z'} + B_{z'} g_2 \, \mathrm{e}^{\delta \xi_2 z'}) \, \mathrm{e}^{\mathrm{i}\delta\zeta x'} \, \mathrm{d}\zeta \tag{37}$$

with

$$\begin{aligned}
A_{x'} &= -g_2\omega'_3 - \omega'_2 g_4 + g_2\omega'_4 + \omega'_2 g_3 - g_3\omega'_4 + \omega'_3 g_4 \\
B_{x'} &= -\omega'_3 g_4 - \omega'_1 g_3 + \omega'_1 g_4 + g_3\omega'_4 + g_1\omega'_3 - g_1\omega'_4 \\
C_{x'} &= g_2\omega'_4 - g_2\omega'_1 + \omega'_1 g_4 - \omega'_2 g_4 + \omega'_2 g_1 - g_1\omega'_4 \\
D_{x'} &= g_2\omega'_1 - g_2\omega'_3 + g_1\omega'_3 - \omega'_1 g_3 + \omega'_2 g_3 - \omega'_2 g_1
\end{aligned} \tag{38}$$

$$\begin{aligned}
A_{z'} &= h_3\omega'_4 - \omega'_3 h_4 - h_2\omega'_4 + \omega'_2 h_4 + \omega'_3 h_2 - h_3\omega'_2 \\
B_{z'} &= h_1\omega'_4 - \omega'_3 h_1 - \omega'_1 h_4 - h_3\omega'_4 + \omega'_3 h_4 + h_3\omega'_1 \\
C_{z'} &= -\omega'_1 h_4 + h_2\omega'_1 - \omega'_2 h_1 + h_1\omega'_4 - h_2\omega'_4 + \omega'_2 h_4 \\
D_{z'} &= \omega'_3 h_2 + \omega'_2 h_1 - h_2\omega'_1 + h_3\omega'_1 - h_3\omega'_2 - \omega'_3 h_1
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
H = {} & -g_2\omega'_3 h_4 + \omega'_3 h_2 g_4 - \omega'_1 g_3 h_4 + g_2\omega'_1 h_4 + \omega'_3 g_2 h_1 + \omega'_2 h_1 g_4 + \omega'_1 g_3 h_2 \\
& + g_2 h_3\omega'_4 - h_3 g_2\omega'_1 - g_2 h_1\omega'_4 - h_2\omega'_1 g_4 + h_3\omega'_1 g_4 - h_3\omega'_2 g_4 - h_1 g_3\omega'_2 \\
& + h_1 g_3\omega'_4 - h_2 g_3\omega'_4 - \omega'_3 h_1 g_4 + \omega'_2 g_3 h_4 - g_1\omega'_3 h_2 - g_1\omega'_2 h_4 + g_1 h_3\omega'_2 \\
& - g_1 h_3\omega'_4 + g_1 h_2\omega'_4 + g_1\omega'_3 h_4
\end{aligned} \tag{40}$$

For a unit concentrated line load applied in the full-space (Green's functions) $\bar{p}_{x'}(\lambda)$ and $\bar{p}_{z'}(\lambda)$ are given by eqn (26) and for a uniform strip load by eqn (29).

It can be shown that the Green's functions present displacement continuity at the interface $z' = 0$ and a stress discontinuity at the origin $x' = 0$, $z' = 0$ where the point load is applied.

There is an alternative approach to obtain Green's functions for a point load applied inside an anisotropic full-space with respect to arbitrarily oriented axes (x', z'). It is possible to synthesize the Green's functions in the rotated coordinate system by merely making a coordinate transformation about θ on the second- and third-order tensors furnished by Rajapakse and Wang (1991) which represent, respectively, displacements and traction kernels in the (x, z) axes of an orthotropic continuum. The expressions for the rotated displacements and stresses are:

$$G_{i'j'} = \alpha_{ik}\alpha_{kl}G_{kl}$$

$$\sigma_{i'j'k'} = \alpha_{il}\alpha_{jm}\alpha_{km}\sigma_{lmn} \tag{41}$$

An analysis of eqns (41) reveals that due to the fact that in the interior of a full-space $G_{zx} = G_{xz}$ (Barros, 1997) then in the rotated system $G_{z'x'} = G_{x'z'}$. Furthermore, all components of $G_{i'j'}$ are symmetric with respect to the point the load is applied $G_{i'j'}(x', z') = G_{i'j'}(-x', -z')$ and all components of the stress functions are antisymmetric with respect to the point the load is applied $\sigma_{i'j'k'}(x', z') = -\sigma_{i'j'k'}(-x', -z')$. It should be stressed that the relations (41) are only valid for a point load applied in the full-space and not for the half-space.

Influence functions for loads applied inside the half-space may also be determined following the methodology outlined in Rajapakse and Wang (1991).

4. Remarks on the numerical realization of the integrals

The solutions presented in the previous sections are in the form of improper integrals that must be evaluated numerically. Two main difficulties are present in the evaluation of these integrals. The first problem is related to the singularities that occur in the integrand and the second is related to the oscillatory integrands and the unbounded character of the integrals.

One situation giving rise to a singular integrand occurs when the values of ω'_i become singular due to the vanishing denominator in eqn (18), $\zeta^2 L_{uxx} + \mathrm{i}\zeta L_{uxz}\xi_i - L_{uzz}\xi_i^2 - \sin\theta = 0$. This is only possible when $\theta = 0$, $\theta = \pi/2$ or when the material is isotropic. For the first two cases the singularities occur for the values $\zeta = \zeta_p$ and $\zeta = \zeta_s$ given by (Rajapakse and Wang, 1991):

$$\zeta_p = \begin{cases} \pm\dfrac{1}{\sqrt{\beta}} & \text{if } \theta = 0 \\ \pm\dfrac{1}{\sqrt{\alpha}} & \text{if } \theta = \pi/2 \end{cases}$$

$$\zeta_s = \pm 1 \tag{42}$$

For an isotropic material $\alpha = \beta = (\bar{\lambda} + 2\bar{\mu})$, where $\bar{\lambda}$ and $\bar{\mu}$ are the Lamé's constants. In this case the value of θ is irrelevant and the singularities are always present.

Another singularity occurs when the load is applied at the surface of the half-space and simultaneously $\theta = 0$ or $\theta = \pi/2$, or when the material is isotropic. This singularity occurs always when $g_1h_2 - g_2h_1 = 0$. For the case $\theta = 0$ the value $\zeta = \zeta_R$ corresponds to the roots of the equation:

$$[2(1-\kappa)\zeta_R^2 - \gamma\zeta_R^2 + \alpha](1-\zeta_R^2) - \alpha\xi_1\xi_2 = 0 \tag{43}$$

For the case $\theta = \pi/2$ the value of α in eqn (43) should be substituted by β. For an isotropic material ζ_R corresponds to the Rayleigh pole. Thus, this singularity is related to Rayleigh waves propagating in the vicinity of the half-space surface. When $0 < \theta < \pi/2$ there are no Rayleigh weaves propagating in an orthotropic material (Synge, 1956). So, for these values of θ, there are no singularities associated with Rayleigh waves in an orthotropic or transversely isotropic material.

To avoid the numerical integration over the singularities associated with ζ_s, ζ_p, ζ_R, material damping has been added to the elastic constants, which become complex and cause the singularities to be displaced from the real integration axes. In the present work a small amount of material damping $v = 0.01$ has been added to the elastic constants c_{11}, c_{33}, c_{13}.

It should be noticed that for orthotropic materials when the principal axes are inclined ($\theta \neq 0$ and $\theta \neq \pi/2$) there are no singularities in the real integration axis associated with ζ and therefore, no addition of material damping is necessary. But to keep the integration procedure homogeneous throughout this article the material damping was added for all the cases.

The inclusion of material damping makes complex the constants T_i of eqn (19). Nevertheless, it has been observed in the numerical simulations that two roots ξ_i still present positive real parts and two possess negative real parts.

Although the addition of material damping removes the singularity from the real axis it is still necessary to treat carefully the integrands in the vicinity of these points because they present very high gradients and a resonance-like behavior. The integration intervals in which these quasi-singular points appear were separated and the numerical integration performed by the routines of the package QUADPACK (Piessens et al., 1983). These routines possess an adaptive procedure for the determination of the integration step increment coupled with the ε-method of series extrapolation (Wynn, 1956) to integrate functions with high gradients. The same series extrapolation method is applied to perform the integration when one of the limits is unbounded. The oscillatory character of the integrand was also treated by the QUADPACK using the Clenshaw–Curtis (Clenshaw and Curtis, 1960) technique which makes the integrals almost insensitive to the value of the product δx (Dravinski and Mossessian, 1988).

5. Numerical results

In this section the numerical results obtained for influence functions applied in the full-space and on the half-space surface are reported. The analysis was conducted for three distinct materials characterized by the elastic constants c_{11}/c_{44}, the anisotropy indexes n_1 and n_3 and the material damping coefficient v, as shown in Table 1. Material 1 is isotropic whereas materials 2 and 3 are

Table 1
Material constants

Material	c_{11}/c_{44}	n_1	n_3	v
1	6	1	1	0.01
2	6	1.5	0.67	0.01
3	6	0.67	1.5	0.01

isotropic. The amplitude of the uniform traction strip load of width $2a$ and centered at the origin of the coordinate system $x'z'$ is p_0. The displacements and stresses are normalized, respectively, by $\bar{u}' = u'c_{44}/(ap_0)$ and $\bar{\sigma} = \sigma/p_0$. All results are for one value of the dimensionless frequency $a_0 = a\omega\sqrt{\rho/c_{44}} = 1$.

5.1. Full-space

Figure 2 shows the displacement component $\bar{u}'_{x'}$ along the axis $z' = 0$ for materials 1, 2 and 3 and for $\theta = 0$. It should be noticed that $\bar{u}'_{z'} = \bar{w}'_{x'} = 0$ along the axis $z' = 0$ when $\theta = 0$ (Rajapakse and Wang, 1991). For the isotropic case, the results of this implementation are compared with the direct integration of the elastic fundamental solution (Manolis and Beskos, 1988) shown as the discrete squares in Fig. 2. The agreement is very good and should be seen as a validation of the present implementation for the isotropic case.

Figures 3 and 4 show the influence of the rotation angle θ on the real and imaginary parts of the normalized displacement components along the axis $z' = 0$ for materials 2 and 3, respectively. It should be noticed that for $\theta \neq 0$, $\bar{u}'_{z'} = \bar{w}'_{x'} \neq 0$. Figure 5 shows the stress component $\bar{\sigma}_{z'x'z'}$ for material 2 and $\theta = \pi/6$. The discontinuities on the real part of the stress distribution, which are in phase with the excitation, are well depicted in the figures. Due to the symmetry of the problem the discontinuities are equally distributed on both sides of the z'-axis. It should be noticed that the depicted discontinuity was obtained numerically. It reproduces very well the given stress boundary condition and can be regarded as a test for the accuracy of the employed numerical integration scheme.

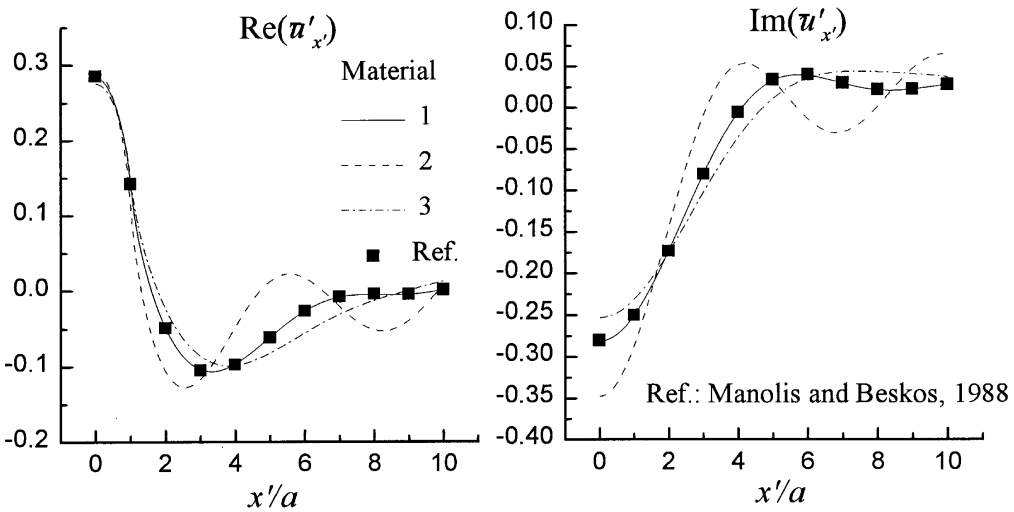

Fig. 2. Normalized displacements along $z' = 0$ due to a uniform strip load applied in the full-space in the x'-direction ($\theta = 0$).

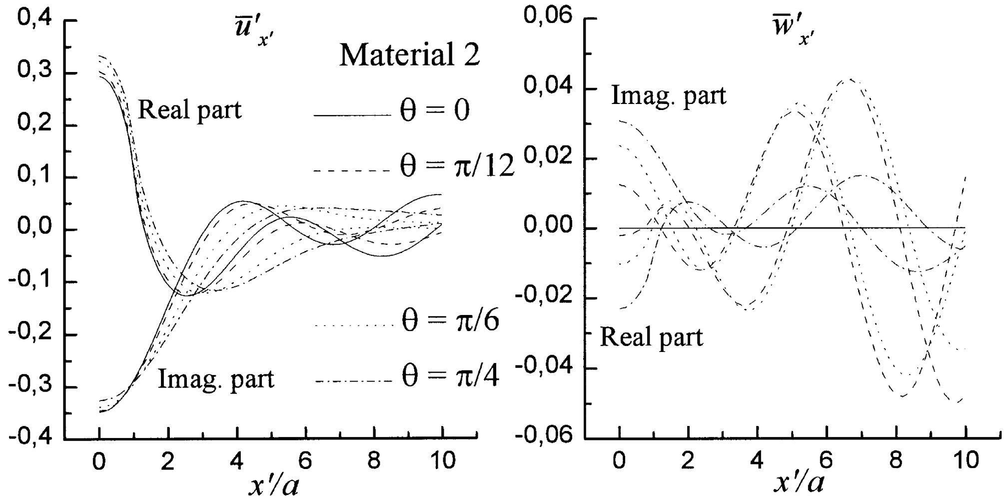

Fig. 3. Normalized displacements along $z' = 0$ due to a uniform strip load applied in the full-space (material 2).

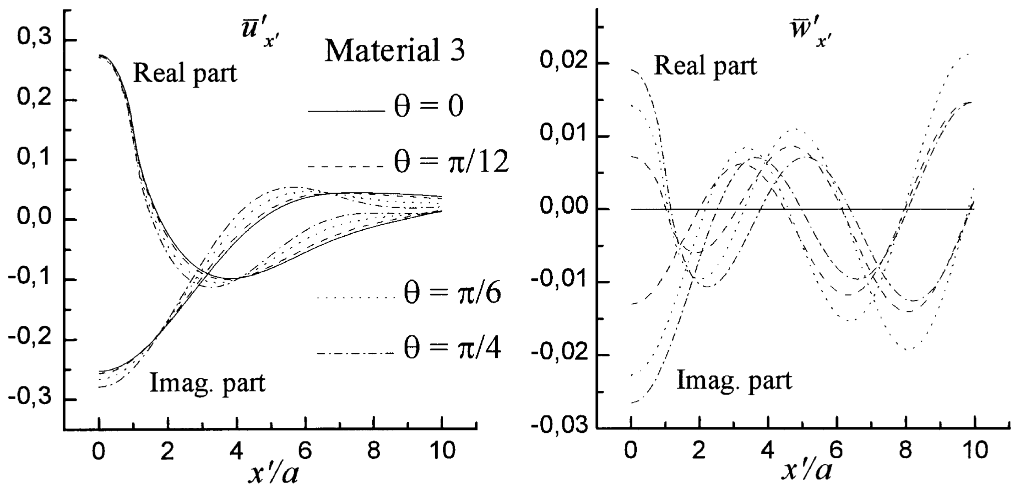

Fig. 4. Normalized displacements along $z' = 0$ due to a uniform strip load applied in the full-space (material 3).

5.2. *Half-space*

Figure 6 shows the displacement components due to a uniform load applied at the surface of a transversely isotropic half-space for materials 1, 2 and 3, considering $\theta = 0$. The discrete points shown at these figures are the results synthesized according to the methodology described by

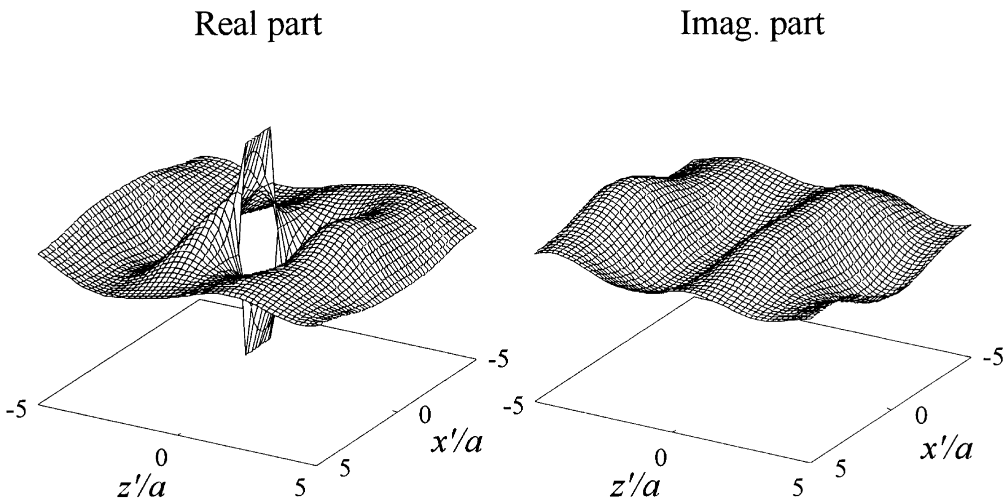

Fig. 5. Normalized stress component $\bar{\sigma}_{x'z'x'}$ in the full-space due to a uniform strip load applied along $z' = 0$ (material 2, $\theta = \pi/6$).

Rajapakse and Wang (1991). It can be seen that the results agree very well, except for the component $\bar{u}'_{x'}$ of the material 2. The discrepancies may be due to differences in the numerical integration procedure. This must be further investigated. But the good agreement in all other components does contribute to validate the present implementation.

The influence of the rotation angle θ on the displacement influence functions at the half-space surface may be seen in Fig. 7 for material 2 and in Fig. 8 for material 3. The rotation angles considered are $\theta = 0$, $\pi/12$, $\pi/6$, $\pi/4$. An analysis of the results reveals that the inclination $\theta \neq 0$ destroys, as expected, the symmetric or anti-symmetric behavior of the components $\bar{u}'_{x'}$ and $\bar{w}'_{x'}$, respectively. It should also be noticed that $\bar{w}'_{x'} \neq \bar{u}'_{z'}$.

The component $\bar{\sigma}_{x'z'x'}$ of the stress tensor in the half-space due to a uniform traction distribution at the surface may be seen in Fig. 9. The results are for material 2 with $\theta = \pi/6$. It can be seen that the numerical procedure reproduces very precisely the prescribed traction boundary condition at the half-space surface. The stress distribution for $\theta \neq 0$ is not symmetric with respect to the z'-axis.

6. Concluding remarks

The article reports a methodology to synthesize the response of plane strain transversely isotropic half-spaces and full-spaces with arbitrarily oriented symmetry axes and subjected to loads and to distributed loads. Numerical results for uniform strip loads, i.e., influence functions, are reported for the half-space surface and for the full-space. The numerical integration strategy is addressed. Validation of the implementation was attempted by comparisons with results reported in the literature for the isotropic continuum and for the transversely isotropic medium with $\theta = 0$. The

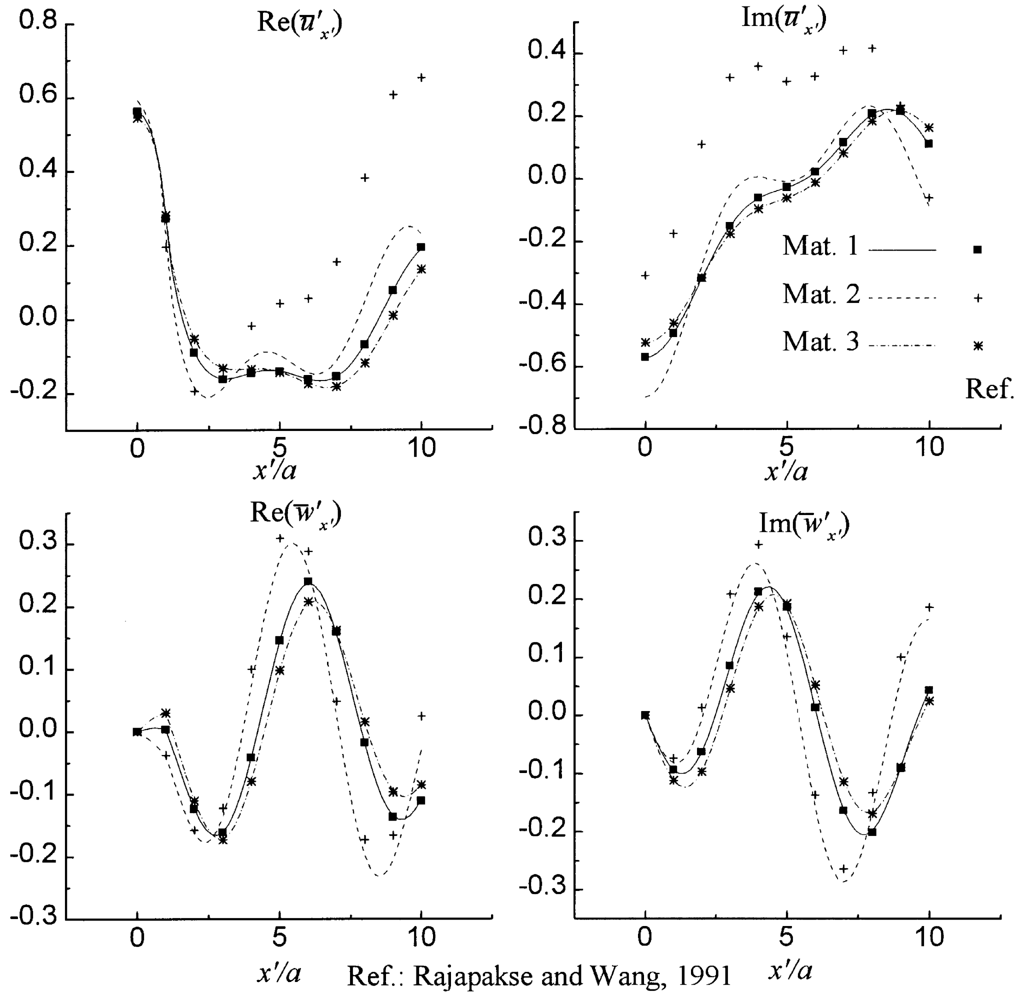

Fig. 6. Normalized displacements along $z' = 0$ due to a uniform strip load applied on the surface of an elastic half-space ($\theta = 0$).

numerical results include two distinct anisotropic states and displays the influence of the rotation angle θ on the continuum displacement and stress responses. The influence functions possess continuous displacements and some discontinuous stress distributions that are accurately reproduced in the present implementation. The methodology presented represents a versatile tool to investigate the influence of the anisotropy ratios (n_1, n_3) and the rotation angle θ as well as the role of the material damping v on the continuum response. It should be noticed that the extension of the present methodology to analyze the more general case of plane anisotropy (Lekhnitskii, 1981) is easily accomplished once the structure of the constitutive equations (4) described in the present article and the one of the plane anisotropy are the same.

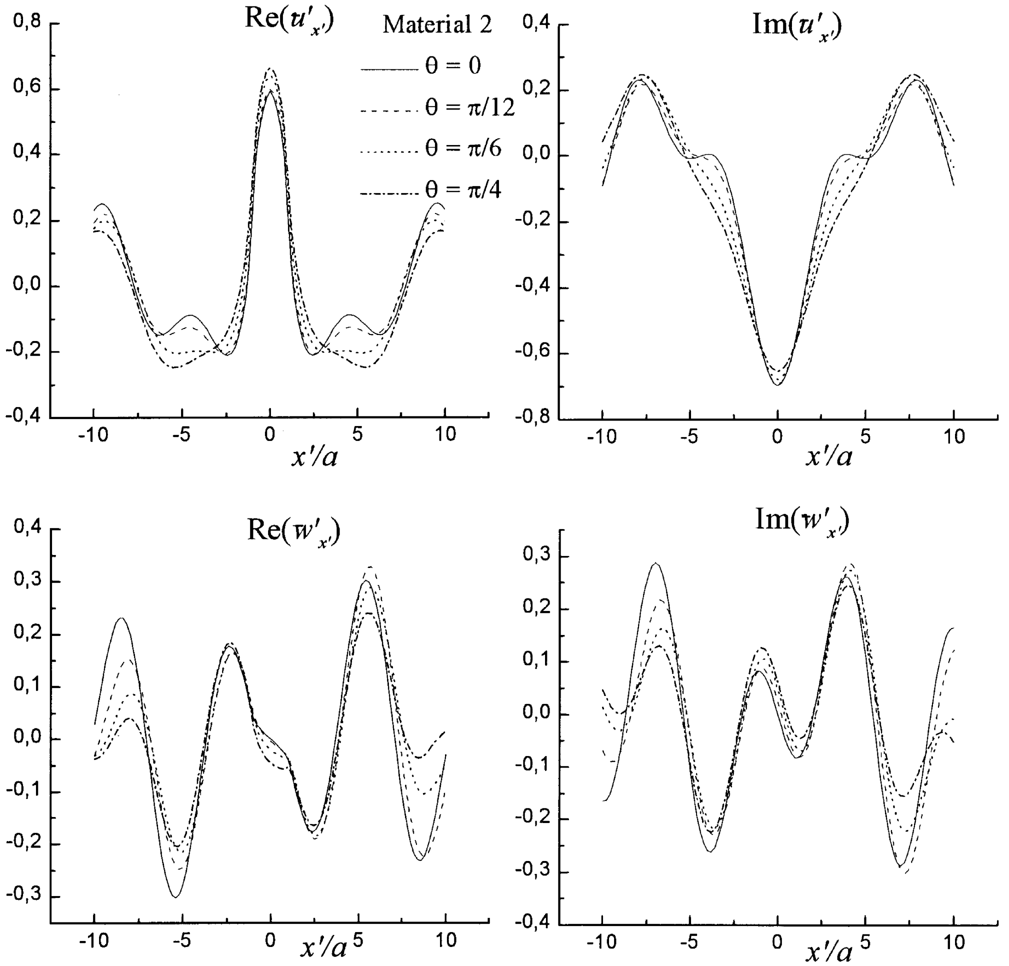

Fig. 7. Normalized displacements along $z' = 0$ due to a uniform strip load applied in the x'-direction at the surface of an elastic half-space (material 2).

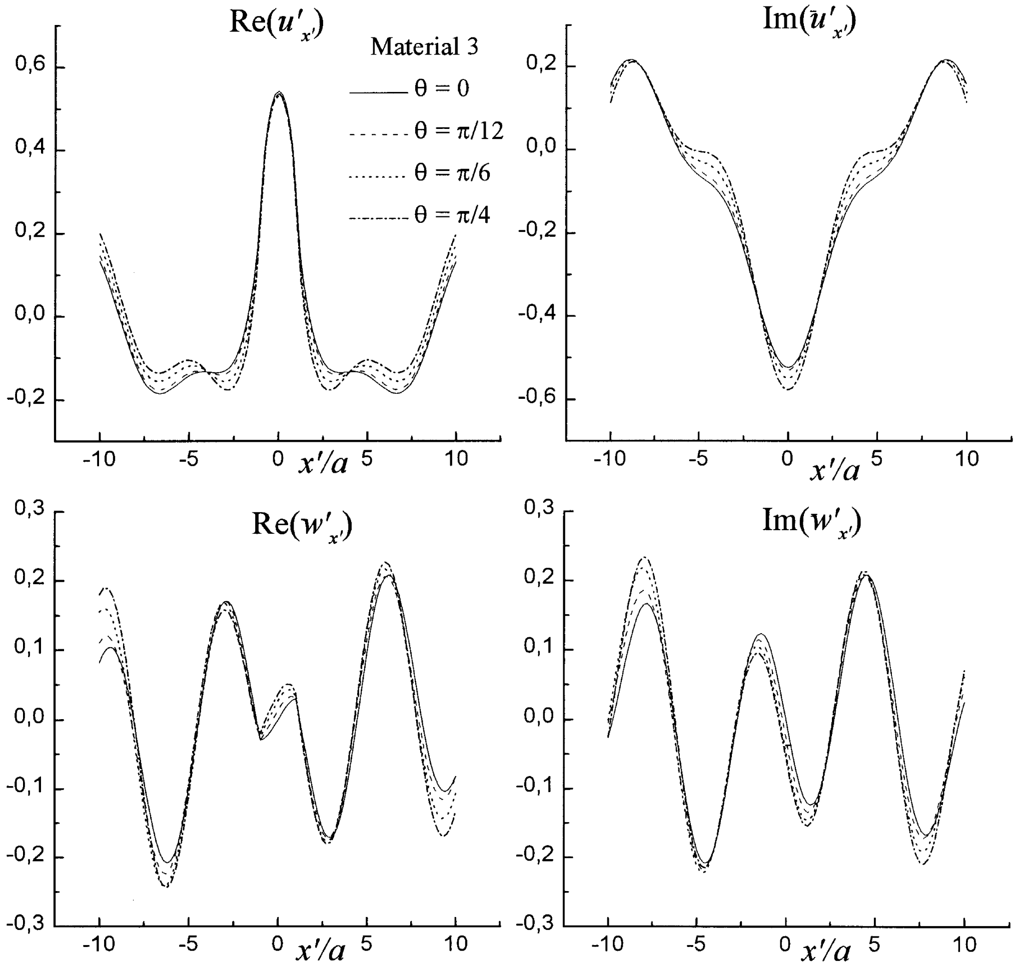

Fig. 8. Normalized displacements along $z' = 0$ due to a uniform strip load applied in the x'-direction at the surface of an elastic half-space (material 3).

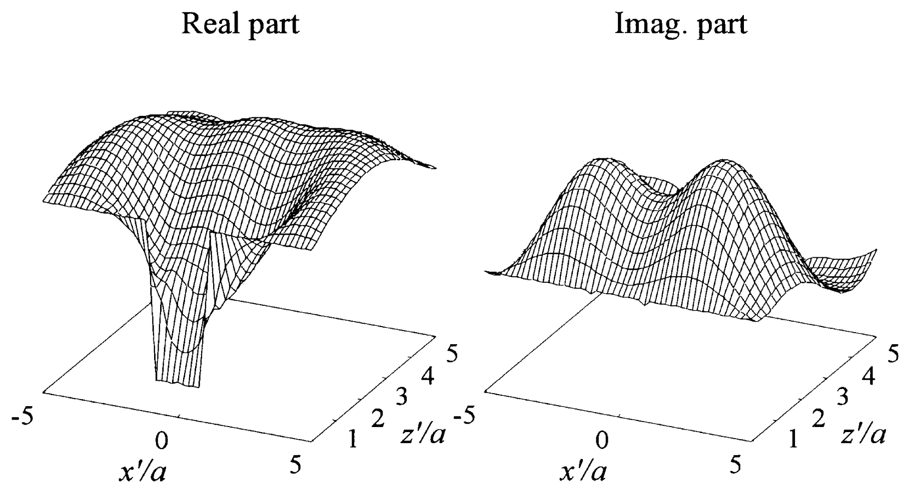

Fig. 9. Normalized stress component $\bar{\sigma}_{x'z'x'}$ within the half-space due to a uniform strip load applied at the surface (material 2, $\theta = \pi/6$).

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